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Combinations of Observables[†]

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This article begins with a review of the framework of fuzzy probability theory. The basic structure is given by the σ-effect algebra of effects (fuzzy events) $\mathscr{E}(\Omega, \mathscr{A})$ and the set of probability measures $M_1^+(\Omega, \mathscr{A})$ on a measurable space $(\Omega, \mathscr{A})$. An observable $X: \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ is defined, where $(\Lambda, \mathscr{B})$ is the value space of X. It is noted that there exists a one-to-one correspondence between states on $\mathscr{E}(\Omega, \mathscr{A})$ and elements of $M_1^+(\Omega, \mathscr{A})$ and between observables $X: \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ and σ-morphisms from $\mathscr{E}(\Lambda, \mathscr{B})$ to $\mathscr{E}(\Omega, \mathscr{A})$. Various combinations of observables are discussed. These include compositions, products, direct products, and mixtures. Fuzzy stochastic processes are introduced and an application to quantum dynamics is considered. Quantum effects are characterized from among a more general class of effects. An alternative definition of a statistical map $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is given.

1. INTRODUCTION

In a recently developed fuzzy probability theory, a crucial role is played by the equivalent concepts of observables (or fuzzy random variables) and statistical maps [1–7, 12, 13]. Most of these previous articles have mainly focused on the properties of statistical maps. The present survey will primarily concentrate on observables and then briefly show how their properties are related to those of statistical maps. Observables offer a viewpoint that has certain advantages. In particular, observables are closer in spirit to the random variables of traditional probability theory and they also are closely related to the quantum observables of operational quantum mechanics [8, 9, 14, 15]. In this brief survey we shall omit proofs and leave them for a later paper.

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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2. FUZZY PROBABILITY THEORY

Let $(\Omega, \mathcal{A})$ be a measurable space. A measurable function $f: \Omega \to [0, 1]$ is called an *effect* or *fuzzy event*. An effect is *crisp* (or *sharp*) if it is an indicator (or characteristic) function. We identify an element $A \in \mathcal{A}$ with its indicator function I_A. In this way, the crisp effects correspond to the events of standard probability theory. The set of effects is denoted by $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{A})$. For $f, g \in \mathcal{E}$, if $f + g \in \mathcal{E}$ (that is, $f + g \leq 1$), then we write $f \perp g$ and define $f \oplus g = f + g$. Denote the set of probability measures on $(\Omega, \mathcal{A})$ by $M_1^+(\Omega, \mathcal{A})$. For $\mu \in M_1^+(\Omega, \mathcal{A})$, we define the *probability* of $f \in \mathcal{E}(\Omega, \mathcal{A})$ by $\mu(f) = \int f\, d\mu$. Notice that $\mu: \mathcal{E}(\Omega, \mathcal{A}) \to [0, 1]$ is a probability measure on $\mathcal{E}(\Omega, \mathcal{A})$ in the following sense. We have $\mu(1) = 1$ and if $f \perp g$, then $\mu(f \oplus g) = \mu(f) + \mu(g)$. Moreover, if $f_i \in \mathcal{E}$ is an increasing sequence, then by the monotone convergence theorem $\mu(\lim f_i) = \lim \mu(f_i)$, so μ is countably additive. Finally, $\mu(I_A) = \mu(A)$ for every $A \in \mathcal{A}$, so μ reduces to the usual probability for crisp effects. It is clear that $(\mathcal{E}, \oplus, 0, 1)$ is an effect algebra [4, 10, 11]. Moreover, if $f_i \in \mathcal{E}$ is an increasing sequence, then $\vee f_i \in \mathcal{E}$, so $\mathcal{E}$ is a σ-effect algebra.

Let P and Q be effect algebras. Recall that a map $\phi: P \to Q$ is a *morphism* if $\phi(1) = 1$ and $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If P and Q are σ-effect algebras, a morphism $\phi: P \to Q$ is a σ-*morphism* if for any increasing sequence $a_i \in P$ we have $\phi(\vee a_i) = \vee \phi(a_i)$. The unit interval $[0, 1] \subseteq \mathbb{R}$ is a σ-effect algebra under the partial operation $a \oplus b = a + b$ whenever $a + b \leq 1$. If $\phi: P \to [0, 1]$ is a σ-morphism, then ϕ is called a *state* on P. It is clear that if $\mu \in M_1^+(\Omega, \mathcal{A})$, then μ is a state on $\mathcal{E}(\Omega, \mathcal{A})$. The following result, which is proved in ref. 13, shows that the set of states on $\mathcal{E}(\Omega, \mathcal{A})$ coincides with $M_1^+(\Omega, \mathcal{A})$.

Theorem 2.1. (i) If $\phi: \mathcal{E}(\Omega, \mathcal{A}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then $\phi(\lambda f) = \lambda \phi(f)$ for every $\lambda \in [0, 1]$. (ii) If $\phi: \mathcal{E}(\Omega, \mathcal{A}) \to [0, 1]$ is a state, then there exists a unique $\mu \in M_1^+(\Omega, \mathcal{A})$ such that $\phi(f) = \mu(f)$ for every $f \in \mathcal{E}(\Omega, \mathcal{A})$.

Let $(\Omega, \mathcal{A})$ and $(\Lambda, \mathcal{B})$ be measurable spaces. A map $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an *observable* on $\mathcal{E}(\Omega, \mathcal{A})$ *with value space* $(\Lambda, \mathcal{B})$ if $X(\Lambda) = 1$ and if $B_i \in \mathcal{B}$, $i \in \mathbb{N}$, are mutually disjoint, then $X(\cup B_i) = \Sigma\, X(B_i)$, where the convergence in the summation is pointwise. We interpret $X(B) \in \mathcal{E}(\Omega, \mathcal{A})$ as the effect or fuzzy event that occurs when X has a value in $B \in \mathcal{B}$. We sometimes use the notation $X(\omega, B) = X(B)(\omega)$. A *probability kernel on* $(\Omega, \mathcal{A})$ *with value space* $(\Lambda, \mathcal{B})$ is a map $K: \Omega \times \mathcal{B} \to [0, 1]$ such that $K(\cdot, B)$ is measurable for every $B \in \mathcal{B}$ and $K(\omega, \cdot) \in M_1^+(\Lambda, \mathcal{B})$ for every $\omega \in \Omega$. Observables and probability kernels are equivalent concepts. Indeed, if $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable, then $X(\omega, B)$ is a probability kernel and conversely, if $K: \Omega \times \mathcal{B} \to [0, 1]$ is a probability kernel, then $X(B)(\omega) =$

$K(\omega, B)$ is an observable. An observable $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is *crisp* (or *sharp*) if $X(B)$ is crisp for every $B \in \mathcal{B}$. If $(\Lambda, \mathcal{B})$ is a Polish measurable space, then it can be shown that $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is crisp if and only if there exists a measurable function $f: \Omega \to \Lambda$ such that $X(B) = I_{f^{-1}(B)}$ for every $B \in \mathcal{B}$ [7]. We use the notation X_f for the crisp observable corresponding to f.

If $\mu \in M_1^+(\Omega, \mathcal{A})$ and $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable, then

$$D_X(\mu) = \mu \circ X \in M_1^+(\Lambda, \mathcal{B})$$

is called the *distribution of X in the state* μ. We interpret $D_X(\mu)(B) = \mu(X(B))$ as the probability that X has a value in B when the system is in the state μ. When $X = X_f$ is crisp, we have $D_X(\mu)(B) = \mu(f^{-1}(B))$, which is the usual distribution of the random variable f. The next result, which is proved in ref. 13, shows that there is a natural one-to-one correspondence between observables and σ-morphisms.

Theorem 2.2. If $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable, then X has a unique extension to a σ-morphism $\tilde{X}: \mathcal{E}(\Lambda, \mathcal{B}) \to \mathcal{E}(\Omega, \mathcal{A})$. If $Y: \mathcal{E}(\Lambda, \mathcal{B}) \to \mathcal{E}(\Omega, \mathcal{A})$ is a σ-morphism, then $Y|\mathcal{B}$ is an observable.

It is shown in ref. 13 that the unique extension $\tilde{X}$ is given by

$$(\tilde{X}g)(\omega) = \int g(\lambda)X(\omega, d\lambda) \tag{2.1}$$

and if X_f is crisp, then $\tilde{X}_f g = g \circ f$. In the sequel, we shall omit the $\sim$ on $\tilde{X}$ and shall frequently identify an observable with its corresponding unique σ-morphism.

Let $(\Omega, \mathcal{A})$, $(\Lambda_1, \mathcal{B}_1)$, and $(\Lambda_2, \mathcal{B}_2)$ be measurable spaces. If $X: \mathcal{B}_1 \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable and $u: \Lambda_1 \to \Lambda_2$ is a measurable function, we define the observable $u(X): \mathcal{B}_1 \to \mathcal{E}(\Omega, \mathcal{A})$ by $u(X)(B) = X(u^{-1}(B))$. We shall see in the next section that $u(X)$ can be viewed as a composition of the observables X and X_u.

3. COMBINATIONS OF OBSERVABLES

Let $(\Omega, \mathcal{A})$, $(\Lambda_1, \mathcal{B}_1)$, and $(\Lambda_2, \mathcal{B}_2)$ be measurable spaces and let $Y: \mathcal{B}_2 \to \mathcal{E}(\Lambda_1, \mathcal{B}_1)$ and $X: \mathcal{B}_1 \to \mathcal{E}(\Omega, \mathcal{A})$ be observables. Although we cannot directly compose X and Y, we can compose them if they are thought of as σ-morphisms.

Doing this, we have the σ-morphism $X \circ Y: \mathcal{E}(\Lambda_2, \mathcal{B}_2) \to \mathcal{E}(\Omega, \mathcal{A})$, which we identify with the observable $X \circ Y: \mathcal{B}_2 \to \mathcal{E}(\Omega, \mathcal{A})$. We call $X \circ Y$ the *composition* of X and Y. We thus have

$$(X \circ Y)(\omega, B) = [X(Y(B))](\omega) = \int Y(B)(\lambda_1)X(\omega, d\lambda_1)$$
$$= \int Y(\lambda_1, B)X(\omega, d\lambda_1) \tag{3.1}$$

which is the usual way of composing probability kernels. We now consider the special cases in which X or Y is crisp. Suppose that Y is crisp and $Y = X_u$, where u: $\Lambda_1 \to \Lambda_2$ is a measurable function. We then have

$$(X \circ X_u)(B) = X(u^{-1}(B)) = u(X)(B)$$

Hence, $X \circ X_u = u(X)$ and $X \circ X_u(\omega, B) = X(\omega, u^{-1}(B))$. Next, suppose that X is crisp and $X = X_f$, where f: $\Omega \to \Lambda_1$ is a random variable. We then have

$$(X_f \circ Y)(B) = (X_f(Y(B)) = Y(B) \circ f$$

and $(X_f \circ Y)(\omega, B) = Y(f(\omega), B)$. Finally, if both X and Y are crisp, we have

$$(X_f \circ X_u)(B) = u(X_f)(B) = f^{-1}(u^{-1}(B)) = (u \circ f)^{-1}(B) = X_{u \circ f}$$

Hence, $(X_f \circ X_u) = X_{u \circ f}$.

Let $(\Omega_i, \mathcal{A}_i)$, $(\Lambda_i, \mathcal{B}_i)$, $i = 1, 2$, be measurable spaces and let X_i: $\mathcal{B}_i \to \mathcal{E}(\Omega_i, \mathcal{A}_i)$, $i = 1, 2$, be observables. Denote the corresponding product spaces by $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$, $(\Lambda_1 \times \Lambda_2, \mathcal{B}_1 \times \mathcal{B}_2)$. Using standard results on product measures, it can be shown that there exists a unique observable

$$X_1 \times X_2\colon \quad \mathcal{B}_1 \times \mathcal{B}_2 \to (\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$$

such that

$$[(X_1 \times X_2)(B_1 \times B_2)](\omega_1, \omega_2) = X_1(B_1)(\omega_1)X_2(B_2)(\omega_2)$$

for all $B_i \in \mathcal{B}_i$, $i = 1, 2$ [6, 12]. We call $X_1 \times X_2$ the *product* of X_1 and X_2. If Y: $\mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{E}(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is an arbitrary observable, then the *marginal observables* Y_i: $\mathcal{B}_i \to (\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$, $i = 1, 2$, for Y are given by $Y_1(B_1) = Y(B_1 \times \Lambda_2)$ and $Y_2(B_2) = Y(\Lambda_1 \times B_2)$. In general, $Y \neq Y_1 \times Y_2$. However, if $Y = X_1 \times X_2$, then the marginal observables for Y are X_1, X_2. This construction can easily be extended to a product $X_1 \times X_2 \times \cdots \times X_n$ of a finite number of observables. More generally, if $(\Omega_t, \mathcal{A}_t)$, $(\Lambda_t, \mathcal{B}_t)$, $t \in T$, are indexed families of measurable spaces, we can form the product spaces $(\times\Omega_t, \times\mathcal{A}_t)$ and $(\times\Lambda_t, \times\mathcal{B}_t)$, where the σ-algebras $\times\mathcal{A}_t$ and $\times\mathcal{B}_t$ are generated by the cylinder sets. We then extend the product construction to form the product $\times X_t$ of observables X_t: $\mathcal{B}_t \to \mathcal{E}(\Omega_t, \mathcal{A}_t)$, $t \in T$.

A similar construction applies for observables X_i: $\mathcal{B}_i \to \mathcal{E}(\Lambda, \mathcal{A})$, $i = 1, 2$, on the same measurable space. In this case, we have the *direct product observable* $X_1 \otimes X_2$: $\mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{E}(\Lambda, \mathcal{A})$, which is the unique observable that satisfies

$$(X_1 \otimes X_2)(B_1 \times B_2) = X_1(B_1)X_2(B_2)$$

for every $B_i \in \mathcal{B}_i$, $i = 1, 2$. Again, we can extend this to direct products $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ of a finite number of observables $X_1, \ldots, X_n$ and to direct products $\otimes X_t$ of an indexed family of observables X_t, $t \in T$.

Let $(\Omega, \mathcal{A})$, $(\Lambda, \mathcal{B})$ be measurable spaces and let $T = \mathbb{R}^+$ or $\mathbb{Z}^+$. Letting $\Lambda_t = \Lambda$ and $\mathcal{B}_t = \mathcal{B}$ for all $t \in T$, we use the notation $\Lambda^T = \times\Lambda_t$ and $\mathcal{B}^T = \times\mathcal{B}_t$, where $\mathcal{B}^T$ is the σ-algebra on Λ^T generated by the cylinder sets. We then form the product space $(\Lambda^T, \mathcal{B}^T)$. The elements $\tilde{\lambda} \in \Lambda^T$ are functions $\tilde{\lambda}: T \to \Lambda$ which we call *paths* in Λ. Recall that $\mathcal{B}^T$ is the smallest σ-algebra on Λ^T such that the projections $\pi_t: \Lambda^T \to \Lambda$ given by $\pi_t(\tilde{\lambda}) = \tilde{\lambda}(t)$ are measurable. We call $\tilde{\lambda}(t)$ *the coordinate of* $\tilde{\lambda}$ *at time* $t \in T$. A *fuzzy stochastic process* is an observable $X: \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$. For $B \in \mathcal{B}$, $t \in T$, we define $B_t \in \mathcal{B}^T$ by

$$B_t = \{\tilde{\lambda} \in \Lambda^T: \tilde{\lambda}(t) \in B\}$$

Then $\{B_t: B \in \mathcal{B}\}$ is a σ-subalgebra of B^T that is isomorphic to $\mathcal{B}$. The observable $X_t: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ defined by $X_t(B) = X(B_t)$ is the *marginal observable for X at time t*. In general, the marginal observables X_t, $t \in T$, do not determine the process X. Conversely, let $X_t: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ be a family of observables, $t \in T$. Then this family generates a fuzzy stochastic process $Y = \otimes X_t$, $Y: \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$ such that $Y_t = X_t$, $t \in T$. However, in general there are other processes with marginals X_t, $t \in T$. If $X: \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$ is crisp (which corresponds to a standard stochastic process), then this ambiguity disappears and $X = \otimes X_t$ [6]. If $X = \otimes X_t$, we call X a *factorizable* fuzzy stochastic process. If X is factorizable, $(\Lambda, \mathcal{B}) = (\Omega, \mathcal{A})$ and $X_{s+t} = X_s \circ X_t$ for all $s, t \in T$, then X is a *Markov process*. In this case, by (3.1) we have

$$X_{s+t}(\omega, A) = X_s \circ X_t(\omega, A) = \int X_t(\omega', A)X_s(\omega, d\omega')$$

which is the Chapman–Kolmogorov equation. A Markov process X for which $T = Z^+$ is called a *Markov chain*. In this case $X_2 = X_1 \circ X_1 = X_1^{(2)}$, $X_3 = X_1 \circ X_2 = X_1^{(3)}, \ldots, X_n = X^{(n)}$.

We can compose a fuzzy stochastic process with an observable to form a new stochastic process. For example, let $X: \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$ be a fuzzy stochastic process and let $Y: \mathcal{A} \to \mathcal{E}(\Lambda', \mathcal{B}')$ be an observable. Then $Y \circ X: \mathcal{B}^T \to \mathcal{E}(\Lambda', \mathcal{B}')$ is the fuzzy stochastic process X *transferred* by Y. As another example, let $Y: \mathcal{B}' \to \mathcal{E}(\Lambda, \mathcal{B})$ be an observable and let $Y_{(t)} = Y$ for every $t \in T$. Then $X \circ (\times Y_{(t)}): \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$ is the process X *pretransferred* by Y. In particular, if $X: \mathcal{A}^T \to \mathcal{E}(\Omega, \mathcal{A})$ gives the evolution of a system and $Y: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable, then $X \circ (\times Y_{(t)}): \mathcal{B}^T \to \mathcal{E}(\Omega, \mathcal{A})$ gives the evolution of Y.

Let $(\Omega, \mathcal{A})$, $(\Lambda, \mathcal{B})$, and $(\Gamma, \mathcal{C})$ be measurable spaces and let $\nu \in M_1^+(\Gamma, \mathcal{C})$. Suppose that $X_\gamma: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$, $\gamma \in \Gamma$, is a collection of observables such that $(\gamma, \omega) \mapsto X_\gamma(\omega, B)$ is measurable for every $B \in \mathcal{B}$. Then for every $B \in \mathcal{B}$

$$\int X_\gamma(B)\nu(d\gamma) \in \mathcal{E}(\Omega, \mathcal{A})$$

is a mixture of the effects $X_\gamma(B)$, $\nu \in \Gamma$. Letting

$$X(B) = \int X_\gamma(B)\nu(d\gamma)$$

it follows from the monotone convergence theorem that $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{A})$ is an observable which we call a *mixture* of X_γ, $\gamma \in \Gamma$. We denote this mixture X by $X = \int X_\gamma \nu(d\gamma)$. By Fubini's theorem, the distribution of X in a state $\mu \in M_1^+(\Omega, \mathcal{A})$ becomes

$$D_X(\mu)(B) = \mu(X(B)) = \int \mu[X_\gamma(B)]\nu(d\gamma)$$

For example, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $f: \Omega \to \mathbb{R}$, $g: \Omega \to \mathbb{R}$ be random variables. Suppose a mixed measurement of f and g is performed according to a ratio $\lambda:(1 - \lambda)$, $0 < \lambda < 1$. Such a measurement yields a distribution $\lambda\mu_f + (1 - \lambda)\mu_g$, where μ_f and μ_g are the distributions of f and g, respectively. In general, no random variable on Ω has this distribution. However, the mixture $X = \lambda X_f + (1 - \lambda)X_g$ is an observable on $\mathcal{E}(\Omega, \mathcal{A})$ that has this distribution in the state μ.

4. QUANTUM OBSERVABLES

Let H be a complex Hilbert space and let $\Omega(H) = \{\omega \in H: \|\omega\| = 1\}$. Endow $\Omega(H)$ with the norm topology τ and let $\mathcal{A}(H)$ be the σ-algebra generated by the open sets in τ. It is well known that the set of linear operators $\mathcal{E}(H)$ on H that satisfy $0 \leq F \leq I$ forms a σ-effect algebra. We now examine the relationship between $\mathcal{E}(H)$ and the σ-effect algebra $\mathcal{E}(\Omega(H), \mathcal{A}(H))$. For $F \in \mathcal{E}(H)$, define $\tilde{F}: \Omega(H) \to [0, 1]$ by $\tilde{F}(\omega) = \langle F\omega, \omega\rangle$. If a sequence $\omega_i \in \Omega(H)$ converges to $\omega \in \Omega(H)$ in the topology τ, then $F^{1/2}\omega_i$ converges to $F^{1/2}\omega$ and hence $\lim\|F^{1/2}\omega_i\|^2 = \|F^{1/2}\omega\|^2$. But

$$\|F^{1/2}\omega\|^2 = \langle F^{1/2}\omega, F^{1/2}\omega\rangle = \langle F\omega, \omega\rangle = \tilde{F}(\omega)$$

and similarly, $\|F^{1/2}\omega_i\|^2 = \tilde{F}(\omega_i)$. Hence, $\lim \tilde{F}(\omega_i) = \tilde{F}(\omega)$, so $\tilde{F}$ is continuous in the τ topology. It follows that $\tilde{F}$ is measurable, so $\tilde{F} \in \mathcal{E}(\Omega(H), \mathcal{A}(H))$. It is easy to show that $\tilde{}: \mathcal{E}(H) \to \mathcal{E}(\Omega(H), \mathcal{A}(H))$ is a σ-morphism. Moreover, if $\tilde{F} \perp \tilde{G}$, then $F \perp G$. It follows that $\tilde{}: \mathscr{E}(H) \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$ is a σ-isomorphism from $\mathscr{E}(H)$ onto the σ-subeffect algebra $\mathscr{E}(H)^{\sim}$ of $\mathscr{E}(\Omega(H), \mathscr{A}(H))$. We call $\mathscr{E}(H)$ and its isomorphic copy $\mathscr{E}(H)^{\sim}$ the set of *quantum effects* on H. Let $(\Lambda, \mathscr{B})$ be a measurable space and let $X: \mathscr{B} \to \mathscr{E}(H)$ be a normalized, positive operator-valued measure [8, 9, 14]. Then $\tilde{X}: \mathscr{B} \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$ defined by $\tilde{X}(B) = X(B)^{\sim}$ is an observable on $\mathscr{E}(\Omega(H), \mathscr{A}(H))$ which we call a *quantum observable*. The distribution $D_{\tilde{X}}(\mu)$ of X for $\mu \in M_1^+(\Omega(H), \mathscr{A}(H))$ becomes

$$D_{\tilde{X}}(\mu)(B) = \mu(\tilde{X}(B)) = \int \tilde{X}(B)(\omega)\mu(d\omega) = \int \langle X(B)\omega, \omega\rangle\mu(d\omega)$$

Since $\mathscr{E}(H)$ and $\mathscr{E}(H)^{\sim}$ are isomorphic, we also call X a *quantum observable*.

It is well known that any state s on $\mathscr{E}(H)$ has the form $s(F) = \mathrm{tr}(FW)$ for a unique positive trace class operator W. By the spectral theorem, W has the unique representation $W = \Sigma\, \lambda_i P_i$, where $\lambda_i > 0$, $\Sigma\, \lambda_i = 1$, and the P_i are mutually orthogonal one-dimensional projections. Let ω_i be unit vectors in the range of P_i, $i = 1, 2, \ldots$, and define the probability measure $\tilde{s}$ on $(\Omega(H), \mathscr{A}(H))$ by $\tilde{s} = \Sigma\, \lambda_i \delta_{\omega_i}$, where δ_ω denotes the Dirac measure concentrated at ω. Then for $F \in \mathscr{E}(H)$ we have

$$\tilde{s}(\tilde{F}) = \sum \lambda_i \tilde{F}(\omega_i) = \sum \lambda_i \langle F\omega_i, \omega_i\rangle = s(F)$$

It follows that if $X: \mathscr{B} \to \mathscr{E}(H)$ is a quantum observable and s is a state on $\mathscr{E}(H)$, then the distribution $B \mapsto s(X(B))$ of X coincides with the distribution $D_{\tilde{X}}(\tilde{s})$ of $\tilde{X}$ relative to $\tilde{s} \in M_1^+(\Omega(H), \mathscr{A}(H))$. In this case, we have

$$D_{\tilde{X}}(\tilde{s})(B) = \sum \lambda_i \langle X(B)\omega_i, \omega_i\rangle = \mathrm{tr}(X(B)W)$$

In particular, $B \mapsto \tilde{X}(B)(\omega) = \langle X(B)\omega, \omega\rangle$ is the distribution of $\tilde{X}$ (and X) in the pure state ω.

We now consider the important question of characterizing the elements of $\mathscr{E}(H)^{\sim}$ in $\mathscr{E}(\Omega(H), \mathscr{A}(H))$. That is, we would like to characterize the effects $f \in \mathscr{E}(\Omega(H), \mathscr{A}(H))$ that are quantum effects. For $f \in \mathscr{E}(\Omega(H), \mathscr{A}(H))$, define $\tilde{f}: H \to \mathbb{R}$ by $\tilde{f}(0) = 0$ and if $\psi \neq 0$, then $\tilde{f}(\psi) = \|\psi\| f(\psi/\|\psi\|)^{1/2}$.

Theorem 4.1. For $f \in \mathscr{E}(\Omega(H), \mathscr{A}(H))$ we have $f \in \mathscr{E}(H)^{\sim}$ if and only if $\tilde{f}$ is a seminorm that satisfies the parallelogram law.

We now consider quantum dynamics. If $U: H \to H$ is a unitary operator, then $U: \Omega(H) \to \Omega(H)$ is continuous and hence measurable. Thus, $X_U: \mathscr{A}(H) \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$ is a crisp observable where, by definition, $X_U(A) = I_{U^{-1}(A)}$. Now suppose that $U(t)$, $t \in \mathbb{R}$, is a dynamical group. That is, $U(t)$ is a unitary operator and $U(s + t) = U(s)U(t)$ for all $s, t \in \mathbb{R}$. For example, $U(t) = e^{-itK}$, the group of unitary transformations generated by the Schrödinger equation, where K is the energy operator. Then $\otimes X_{U(t)}\colon \mathscr{A}(H)^{\mathbb{R}} \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$ is a fuzzy stochastic process. If $Y\colon \mathscr{B} \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$ is an observable, then

$$\otimes X_{U(t)} \circ \times Y_{(t)}\colon \quad \mathscr{B}^{\mathbb{R}} \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$$

describes the evolution of Y. To verify this statement for quantum observables, let $X\colon \mathscr{B} \to \mathscr{E}(H)$ be a quantum observable. Then in conventional quantum mechanics, $U(t)^*XU(t)$, $t \in \mathbb{R}$, describes the evolution of X. Then for every $t \in \mathbb{R}$, we have

$$[U(t)^*XU(t)]^\sim\colon \quad \mathscr{B} \to \mathscr{E}(\Omega(H), \mathscr{A}(H))$$

and

$$\otimes[U(t)^*XU(t)]^\sim\colon \quad \mathscr{B}^{\mathbb{R}} \to \mathscr{E}(\Omega\,(H), \mathscr{A}(H))$$

is a fuzzy stochastic process. The next result shows that this process is given by our previous description.

Theorem 4.2. In terms of our previous notation, we have

$$\otimes \tilde{X}_{U(t)} \circ \times \tilde{X}_{(t)} = \otimes[U(t)^*XU(t)]^\sim$$

5. STATISTICAL MAPS

We now discuss the relationship between observables and statistical maps. If $(\Omega, \mathscr{A})$, $(\Lambda, \mathscr{B})$ are measurable spaces, a function $f\colon \Omega \to M_1^+(\Lambda, \mathscr{B})$ called a *fuzzy random variable* (or a *statistical function*) [1–7] if $\omega \mapsto [f(\omega)](B)$ is measurable for every $B \in \mathscr{B}$. There is a one-to-one correspondence between observables and fuzzy random variables. Indeed, if $X\colon \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ is an observable, then $\hat{X}\colon \Omega \to M_1^+(\Lambda, \mathscr{B})$ defined by $\hat{X}(\omega)(B) = X(\omega, B)$ is a fuzzy random variable. Conversely, if $f\colon \Omega \to M_1^+(\Lambda, \mathscr{B})$ is a fuzzy random variable, then $f^\vee\colon \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ defined by $f^\vee(B)(\omega) = f(\omega)(B)$ is an observable.

A map $T\colon M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is *measurable* if $\omega \mapsto (T\delta_\omega)(B)$ is measurable for every $B \in \mathscr{B}$. A weak topology on $M_1^+(\Omega, \mathscr{A})$ $[M_1^+(\Lambda, \mathscr{B})]$ is induced by the weak topology resulting from the duality between measures and bounded measurable functions on $(\Omega, \mathscr{A})$ $[(\Lambda, \mathscr{B})]$. We call $T\colon M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ a *statistical map* if T is affine, measurable, and weakly continuous. If $X\colon \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ is an observable, it is clear that its distribution map $D_X\colon M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is affine. The next result shows that D_X is a statistical map and that every statistical map has this form.

Theorem 5.1. A map $T\colon M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is a statistical map if and only if there exists an observable $X\colon \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ such that $T = D_X$. Moreover, X is unique.

Corollary 5.2. If $X, Y: B \to \mathscr{E}(\Omega, \mathscr{A})$ are observables for which $D_X = D_Y$, then $X = Y$.

Corollary 5.3. If $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is a statistical map, then for every $\mu \in M_1^+(\Omega, \mathscr{A})$ and $B \in \mathscr{B}$ we have

$$(T\mu)(B) = \int (T\delta_\omega)(B)\mu(d\omega) \tag{5.1}$$

Conversely, if $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is measurable and satisfies (5.1), then T is a statistical map.

In refs. 1–7 a statistical map is defined to be a measurable map $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ that satisfies (5.1). (It is also assumed that T is affine, but this condition is redundant.) Corollary 5.3 shows that this definition is equivalent to the one we have given. However, we believe that our definition is more basic and easier to verify.

We have seen that if $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is a statistical map, then there exists a unique observable $X: \mathscr{B} \to \mathscr{E}(\Omega, \mathscr{A})$ such that $T = D_X$. We say that T is *crisp* if X is crisp. Thus, T is crisp if and only if there exists a random variable $f: \Omega \to \Lambda$ such that $T = D_{X_f}$. In this case $T\mu = \mu_f$, the distribution of f relative to μ. Denote the set of δ measures on $(\Omega, \mathscr{A})$ by $\partial M_1^+(\Omega, \mathscr{A})$. The proof of the following result is outlined in ref. 7.

Theorem 5.4. A statistical map $T: M_1^+(\Omega, \mathscr{A}) \to M_1^+(\Lambda, \mathscr{B})$ is crisp if and only if $T[\partial M_1^+(\Omega, \mathscr{A})] \subseteq \partial M_1^+(\Lambda, \mathscr{B})$.

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